B.sc(H) part 1 paper 1

Topic:Inverse circular &Hyperbolic Functions of Complex Quantitie

Subject:Mathematics

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1: Inverse circular functions of complex quantities

If $\sin(x + iy) = \alpha + i\beta$, then $x + iy$ is said to be inverse sine of $(\alpha + i\beta)$ and is denoted by $\sin^{-1}(\alpha + i\beta)$.

Thus $$x + iy = \sin^{-1}(\alpha + i\beta) \quad ...(1)$$

We have also,

$$\sin\{n\pi + (-1)^n(x + iy) = \sin(x + iy) = \alpha + i\beta$$

$$\therefore \quad n\pi + (-1)^n(x + iy) = \sin^{-1}(\alpha + i\beta) \quad ...(2$$

Similarly if $\cos(x + iy) = \alpha + i\beta$, then

$$\cos^{-1}(\alpha + i\beta) = 2n\pi \pm (x + iy)$$

or if $\tan(x + iy) = \alpha + i\beta$, then $\tan^{-1}(\alpha + i\beta) = n\pi + (x + iy)$ etc.

These results follow from the following considerations which we have done earlier.

If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$;

if $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$;

if $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$.

Thus we find that the inverse circular function of a complex quantity is a many-valued function.

The function thus obtained may be described as the general value of the inverse function.

The principal value of the inverse function is obtained by putting $n = 0$ in the general value of the function and is defined as follows :

The principal value of $\sin^{-1}(\alpha + i\beta)$ is that value of $n\pi + (-1)^n(x + iy)$ (i.e. $x + iy$, on putting $n = 0$) whose real part x lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

The principal value of $\cos^{-1}(\alpha + i\beta)$ is that value of $2n\pi \pm (x + iy)$ which is such that its real part x (on putting $n = 0$) lies between 0 and π.

The principal value of $\tan^{-1}(\alpha + i\beta)$ is that value of $n\pi + (x + iy)$ which is such that its real part x lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

2:Inverse Hyperbolic Functions

These are defined in the same manner as inverse circular functions.

If $\sin hz = w$, then z is called the inverse hyperbolic sine of w and is denoted by $\sin h^{-1}w$ and we write $z = \sin h^{-1}w$.

Similarly, we define $\cos h^{-1}w$, $\tan h^{-1}w$ etc.

Case I. Let $w = \sin hz$

where w and z both are imaginary numbers given by $w = u + iv$ and $z = x + iy$, then $z = \sin h^{-1}w$.

Now, $$w = \sin hz = \frac{1}{2}(e^z - e^{-z}) = \frac{1}{2}\left(e^z - \frac{1}{e^z}\right)$$

$\therefore$ $$e^{2z} - 2we^z - 1 = 0.$$

Treating this as a quadratic in e^z, we get

$$e^z = \frac{2w \pm \sqrt{4w^2 + 4}}{2} = w \pm \sqrt{w^2 + 1}$$

$\therefore$ $$z = 2n\pi i + \log(w \sqrt{w^2 + 1})$$

or, $$z = 2n\pi i + \log(w - \sqrt{w^2 + 1}).$$

Since $$w - \sqrt{w^2 + 1} = \frac{(w - \sqrt{w^2 + 1}) \times (w + \sqrt{w^2 + 1})}{(w + \sqrt{w^2 + 1})}$$

$$= \frac{-1}{w + \sqrt{w^2 + 1}}$$

$\therefore$ $$\log(w - \sqrt{w^2 + 1}) = \log(-1) - \log(w + \sqrt{w^2 + 1})$$

$$= i\pi - \log(w + \sqrt{w^2 + 1}).$$

Thus, $$z = 2n\pi i + \log(w + \sqrt{w^2 + 1})$$

or, $$z = (2n + 1)i\pi - \log(w + \sqrt{w^2 + 1})$$

$$= (2n + 1)i\pi + (-1)^{2n+1}\log(w + \sqrt{w^2 + 1}).$$

Both the values of z can be included in the expression

$$z = n\pi i + (-1)^n \log (w + \sqrt{w^2 + 1})$$

which is the general value of $\sin h^{-1} w$.

The principal value of $\mathbf{\sin h^{-1} w = \log (w + \sqrt{w^2 + 1})}$.

Case II. Let $w = \cos hz$, then $z = \cos h^{-1} w$.

Now $w = \cos hz = \dfrac{1}{2}(e^z + e^{-z}) = \dfrac{1}{2}\left(e^z + \dfrac{1}{e^z}\right)$.

$\therefore \quad e^{2z} - 2we^z + 1 = 0.$

As before, $\quad e^z = \dfrac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1}.$

Hence $z = \log (w \pm \sqrt{w^2 - 1})$.

Since $w - \sqrt{w^2 - 1} = \dfrac{(w - \sqrt{w^2 - 1}) \times (w + \sqrt{w^2 - 1})}{w + \sqrt{w^2 - 1}}$

$$= \frac{1}{w + \sqrt{w^2 - 1}}$$

$\therefore \quad \log (w - \sqrt{w^2 - 1}) = - \log (w + \sqrt{w^2 - 1}).$

Thus $z = 2n\pi i \pm \log (w + \sqrt{w^2 - 1})$ which is the general value of $\cos h^{-1} w$, and the principal value of

$$\mathbf{\cos h^{-1} w = \log (w + \sqrt{w^2 - 1})}.$$

Case III. Let $w = \tan hz$ so that $z = \tan h^{-1} w$.

Now $\quad w = \tan hz = \dfrac{\sin hz}{\cos hz} = \dfrac{e^z - e^{-z}}{e^z + e^{-z}} = \dfrac{e^{2z} - 1}{e^{2z} + 1}.$

$\therefore \quad e^{2z} = \dfrac{1 + w}{1 - w}$ (by componendo and dividendo)

$\Rightarrow \quad 2z = 2n\pi i + \log \dfrac{1 + w}{1 - w}$

$\Rightarrow \quad z = n\pi i + \dfrac{1}{2} \log \dfrac{1 + w}{1 - w}$

which is the general value of $\tan h^{-1} w$.

The principal value of $\mathbf{\tan h^{-1} w = \dfrac{1}{2} \log \dfrac{1 + w}{1 - w}}$.

Similarly the general and principal values of $\operatorname{cosec} h^{-1} w$, $\sec h^{-1} w$ and $\cot h^{-1} w$ may be obtained.

4.8 Relation between the Inverse Hyperbolic Functions and Inverse Circular Functions

If $\quad w = \sin hz$ ...(1)

then $w = -i \sin (iz)$, since $\sin hz = -i \sin (iz)$

$\Rightarrow \quad iw = \sin iz$ ...(2)

From (1), $z = \sin h^{-1} w$,

and from (2), $iz = \sin^{-1} (iw) \Rightarrow z = -i \sin^{-1}(iw)$

Hence $\quad \sin h^{-1} w = -i \sin^{-1}(iw)$.

Similarly, $\quad \cos h^{-1} w = -i \cos^{-1}(w)$,

and $\quad \tan h^{-1} w = -i \tan^{-1}(iw)$, and so on.